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PROBLEMS OF THE CONCENTRATION OF ELASTIC STRESSES NEAR DEFECTS IN SPHERICALLY MULTILAYERED MEDIA†

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A spherically multilayered medium, whose elastic parameters change abruptly on the spherical surfaces, with defects in the form of cracks or thin rigid inclusions, is considered. The method of solving problems of the stress concentration near such defects is based on the introduction of linear combinations of the displacements and stresses as the fundamental unknowns. This enables the difficulties related to the presence of an arbitrary number of layers to be effectively overcome. The method is described initially for an unbounded elastic medium and defects of spherical form, situated on the surfaces where the elastic parameters change (interphase defects) and a way of extending this to the case of an elastic medium of finite dimensions, defects of other forms and not situated on these surfaces, is indicated. The method is described in detail as it applies to the case of a two-layer medium with an interphase crack when a torsion centre at the origin of coordinates acts on the medium. The problem is reduced to an integral equation, an effective method of solving it is given, and a formula is obtained for the stress intensity factor.

Axisymmetric problems of stress concentration in a two-layer medium with a crack were considered earlier in [1, 2].

1. THE INTRODUCTION OF NEW UNKNOWN FUNCTIONS AND THE METHOD OF FINDING THEIR TRANSFORMANT

We will denote the components of the displacement field $u_r = u_r(r, \theta, \varphi)$, $u_\theta = u_\theta(r, \theta, \varphi)$, $u_\varphi = u_\varphi(r, \theta, \varphi)$ as follows: $2G[u_r, u_\theta, u_\varphi] = [u, \upsilon, w]$ (G and μ are the shear modulus and Poisson's ratio) and we will conventionally indicate a partial derivative with respect to r by a prime, a derivative with respect to θ by a dot and a derivative with respect to φ by a comma. Instead of the displacements υ and w we will introduce the new unknowns $z(r, \theta, \varphi)$ and $z^*(r, \theta, \varphi)$, and instead of the shear stresses $\tau_{r\theta} \equiv \tau_\theta$ and $\tau_{r\varphi} \equiv \tau_\varphi$ we will introduce the function $\tau(r, \theta, \varphi)$ and $\tau^*(r, \theta, \varphi)$ by the formulae

$$\sin\theta\begin{Vmatrix} z \\ z^* \end{Vmatrix} = \begin{Vmatrix} \upsilon \\ w \end{Vmatrix}\sin\theta\Bigg\|^{\cdot} \pm \begin{Vmatrix} w \\ \upsilon \end{Vmatrix}^{,}, \quad \sin\theta\begin{Vmatrix} \tau \\ \tau^* \end{Vmatrix} = \begin{Vmatrix} \tau_\theta \\ \tau_\varphi \end{Vmatrix}\sin\theta\Bigg\|^{\cdot} \pm \begin{Vmatrix} \tau_\varphi \\ \tau_\theta \end{Vmatrix}^{,} \tag{1.1}$$

Here the Lamé equations, written in a spherical system of coordinates [3], are separated into an harmonic equation for z^* and a system of two equations for u and z. In order to simplify the search for the functions introduced above, we will change to Fourier transformants

$$[u_n(r,\theta), z_n(r,\theta), z_n^*(r,\theta)] = \int_{-\pi}^{\pi} \frac{[u(r,\theta,\varphi), z(r,\theta,\varphi), z^*(r,\theta,\varphi)]}{e^{in\varphi}} d\varphi, \; n = 0, \pm 1, \pm 2, \ldots \tag{1.2}$$

and Legendre transformants ($P_k^m(z)$ is the associated Legendre function)

$$[u_{nk}(r), z_{nk}(r), z_{nk}^*(r)] = \int_0^{\pi} \frac{P_k^{|n|}(\cos\theta)[u_n(r,\theta), z_n(r,\theta), z_n^*(r,\theta)]}{\operatorname{cosec}\theta} d\theta, \; k = 0, 1, 2, \ldots \tag{1.3}$$

for which we know the inversion formulae

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$$u(r,\theta,\varphi)=\sum_{n=-\infty}^{\infty}\frac{u_n(r,\theta)}{e^{-in\varphi}},\quad u_n(r,\theta)=\sum_{k=|n|}^{\infty}\sigma_{kn}u_{nk}(r)P_k^{|n|}(\cos\theta),\quad \sigma_{kn}=\frac{(k-|n|)!}{(k+|n|)!}\frac{2k+1}{2} \tag{1.4}$$

It can be shown that the Fourier transformants of the stresses can be expressed in terms of the new unknowns as

$$\begin{aligned}&(1-2\mu)\sigma_{rn}(r,\theta)=(1-\mu)u_n'(r,\theta)+\mu r^{-1}[2u_n(r,\theta)+z_n(r,\theta)]\\ &2r\tau_n(r,\theta)=r^2(r^{-1}z_n)'-\nabla_n u_n,\quad 2r\tau_n^*(r,\theta)=r^2(r^{-1}z_n^*)'\\ &\nabla_n f(r,\theta)=(\sin\theta)^{-2}n^2 f(r,\theta)-(\sin\theta)^{-1}[\sin\theta f^{\bullet}(r,\theta)]^{\bullet}\end{aligned} \tag{1.5}$$

As a consequence of the fact that the function z^* is harmonic, its Fourier–Legendre transformant $z^*_{nk}(r)$ will, in general, be defined by the formula

$$z_{nk}^*(r)=X_{nk}r^k+Y_{nk}r^{-k-1},\quad k=0,1,2,\ldots,\quad n=0,\pm1,\pm2,\ldots \tag{1.6}$$

where X_{nk} and Y_{nk} are arbitrary constants.

In order to obtain similar general representations for $u_{nk}(r)$ and $z_{nk}(r)$, it is convenient to start from the formulae obtained by Lamé [4] for the displacements u_r, u_θ, u_φ and written by Ulitko [5] in the form

$$u_r(r,\theta,\varphi)=\sum_{k=n}^{\infty}\sum_{n=-k}^{k}\frac{P_k^{|n|}(\cos\theta)u_k^{(n)}(r)}{\sqrt{2\pi\sigma_{kn}}e^{-in\varphi}} \tag{1.7}$$

$$u_k^{|n|}(r)=\mu_k^+A_k^{(n)}r^{k+1}+B_k^{(n)}r^{k-1}+\mu_k^-C_k^{(n)}r^{-k}-D_k^{(n)}r^{-k-2}$$

$$\mu_k^+=k-2-4\mu,\quad \mu_k^-=k+3-4\mu$$

Here $A_k^{(n)}$, $B_k^{(n)}$, $C_k^{(n)}$, $D_k^{(n)}$ are arbitrary constants. Changing the order of summation in the double series (1.7) we apply integral transformations (1.2) and (1.3) to it. As a result we obtain the equation $u_{nk}(r)=2G(2\pi\sigma_{kn})^{-1/2}u_k^{(n)}(r)$, and hence

$$u_{nk}(r)=\mu_k^+X_{nk}^0r^{k+1}+X_{nk}^1r^{k-1}+\mu_k^-Y_{nk}^0r^{-k}-Y_{nk}^1r^{-k-2} \tag{1.8}$$

where $X_{nk}^{0,1}$, $Y_{nk}^{0,1}$ are new arbitrary constants. Carrying out similar operations on the formulae for u_θ and u_φ from [5], taking formulae (1.1) into account, in the transformants having the form

$$\sin\theta\begin{Vmatrix}z_n\\ z_n^*\end{Vmatrix}=\frac{\partial}{\partial\theta}\begin{Vmatrix}v_n\\ \\ w_n\end{Vmatrix}\sin\theta\begin{Vmatrix}\\ \\ \end{Vmatrix}\pm in\begin{Vmatrix}w_n\\ v_n\end{Vmatrix} \tag{1.9}$$

we obtain

$$-z_{nk}(r)=k\mu_{k+2}^-X_{nk}^0r^{k+1}+(k+1)X_{nk}^1r^{k-1}-(k+1)\mu_{k-2}Y_{nk}^0r^{-k}+kY_{nk}r^{-k-2} \tag{1.10}$$

After applying transformation (1.2) to the second formula from (1.1) and applying transformation (1.3) to formulae (1.4) and subsequently using (1.8) and (1.10) we obtain

$$\begin{aligned}\sigma_{rnk}&=\mu_k^\sigma X_{nk}^0r^k+(k-1)X_{nk}^1r^{k-2}-\mu_{k+2}^\sigma Y_{nk}^0r^{-k-1}+(k+2)Y_{nk}^1r^{-k-3}\\ -\tau_{nk}&=k\mu_{k+1}^\tau X_{nk}^0r^k+(k^2-1)X_{nk}^1r^{-k-2}+(k+1)\mu_k^\tau Y_{nk}^0r^{-k-1}\\ &-k(k+2)Y_{nk}^1r^{-k-3}(\mu_k^\sigma=k(k-1)-2-2\mu,\quad \mu_k^\tau=k^2-2+2\mu)\\ 2\tau_{nk}^*&=X_{nk}(k-1)r^{k-1}-Y_{nk}(k+2)r^{-k-2}\end{aligned} \tag{1.11}$$

We will use the relations obtained to solve this problem. The elastic medium fills the outside of a spherical cavity of radius R. Shear stresses are applied to the surface of this cavity, i.e.

$$\tau^0_{r\varphi}|_{r=R} = \tau^0_{\varphi}|_{r=R} = A\sin\theta, \quad 0 \leqslant \theta \leqslant \pi \tag{1.12}$$

It is required to find the stresses and displacements. Since there is axial symmetry, we must put $n = 0$ in all the previous formulae, since by (1.2) $\tau_{\varphi 0}(r, \theta) = \tau_{r\varphi}$, $w_0(r, \theta) = w(r, \theta)$. Using the second formula of (1.1), written in terms of transformants, and taking (1.12) into account, we find that $\tau^*_0(R, \theta) = 2A\cos\theta$ and correspondingly (δ_{kj} is the Kronecker delta)

$$\tau^*_{0k}(R) = 4A(2k+1)^{-1}\delta_{k1} \tag{1.13}$$

If we construct a solution that is regular at infinity, then, in formula (1.11) for $\tau^*_{nk}(r)$ we must put $X_{0k} = 0$ and obtain Y_{0k} from condition (1.13); we will thereby find the transformants $\tau^*_{0k}(r)$ and $z^*_{0k}(r)$. Then, using the corresponding inversion formula (1.4), we finally obtain

$$\tau^*_0(r,\theta) = \frac{3M}{4\pi r^3}\cos\theta, \quad z^*_0(r,\theta) = -\frac{M\cos\theta}{2\pi r^2}, \quad M = \frac{8\pi AR^3}{3} \tag{1.14}$$

It can be shown that M is the torque produced by the shear stresses (1.12).

If we now allow R to approach zero and the constant A to approach infinity, so that the torque remains unchanged and equal to the specified M, formulae (1.14) give the stress field and the displacements from the torsion centre at the origin of coordinates.

As can be seen, the new functions introduced can be found fairly simply. By determining them the functions υ_n and w_n can be found as follows. Using the obvious linear combination of Eqs (1.19), we obtain differential equations for υ_n and w_n, which differ solely in the right-hand sides and can be solved simply using integral transformation (1.3). As a result, we arrive at the formulae

$$\left\|\begin{matrix} \upsilon_n(r,\theta) \\ w_n(r,\theta) \end{matrix}\right\| = -\int_0^\pi \sin t\, \Phi_n(\theta,t)\left[\frac{\partial}{\partial t}\left\|\sin^2 t \begin{matrix} z_n(r,t) \\ z^*_n(r,t) \end{matrix}\right\| \mp in \left\|\begin{matrix} z^*_n(r,t) \\ z_n(r,t) \end{matrix}\right\|\right]dt$$

$$\Phi_n(\theta,t) = \sum_{k=|n|}^{\infty} \frac{\sigma_{kn}}{k(k+1)} P_k^{|n|}(\cos\theta) P_k^{|n|}(\cos t) \tag{1.15}$$

This formula is unsuitable when $n = 0$, i.e. for axisymmetric problems, but in this case, putting $n = 0$ in (1.9), we can obtain the simpler formulae

$$\left\|\begin{matrix} \upsilon_0(r,\theta) \\ w_0(r,\theta) \end{matrix}\right\| = \frac{1}{\sin\theta}\int_0^\theta \left\|\begin{matrix} z_0(r,t) \\ z^*_0(r,t) \end{matrix}\right\| \sin t\, dt \tag{1.16}$$

2. THE REDUCTION OF PROBLEMS OF STRESS CONCENTRATION IN SPHERICALLY MULTI-LAYERED MEDIA TO A SYSTEM OF EQUATIONS AND AN EFFECTIVE METHOD OF SOLVING IT

Consider the following problem. In an unbounded spherically multilayered elastic medium, arbitrarily loaded by body forces, there are defects in the form of cracks or thin inclusions, situated on the spherical surfaces where the elasticity constants change. It is required to determine the stress and displacement distribution in such a medium.

We will denote the radii of the spherical surfaces on which sudden jumps in the elasticity constants occur by $R_i (i = 0, 1, 2, \ldots m)$ so that when $R_{i-1} < r < R_i$ Poisson's ratio and the shear modulus take values μ_i and G_i, where $R_{-1} = 0$, $R_{m+1} = \infty$. We take as the fundamental unknowns the functions introduced above, the Fourier–Legendre transformants of which are given by (1.6), (1.8), (1.10) and (1.11), and each layer has its own arbitrary constants ${}_iX^{0,1}_{nk}$, ${}_iY^{0,1}_{nk}$ and elasticity parameters μ_i and G_i. For example, for $u_{nk}(r)$ we have the formula

$$\begin{gathered} {}_iu_{nk}(r) = {}_iX^0_{nk}r^{k+1}{}_i\mu^+_k + {}_iX^1_{nk}r^{k-1} + {}_iY^0_{nk}r^{-k}{}_i\mu^-_k - {}_iY_{nk}r^{-k-2} + {}_iu^0_{nk}(r) \\ {}_i\mu^+_k = k-2+4\mu_i, \quad {}_i\mu^-_k = k+3-4\mu_i, \quad i = 0,1,\ldots, \quad R_{i-1} < r < R_i \end{gathered} \tag{2.1}$$

Similar formulae exist for the remaining transformants

$$ {}_iz_{nk}(r),\quad {}_iz^*_{nk}(r),\quad {}_i\sigma_{rnk}(r),\quad {}_i\tau_{nk}(r),\quad {}_i\tau^*_{nk}(r) \tag{2.2}$$

Here, in order to ensure that these transformants are regular at zero and at infinity, we must put

$$ {}_0Y_{nk} = {}_0Y^0_{nk} = {}_0Y^1_{nk} = 0,\quad {}_{m+1}X_{nk} = {}_{m+1}X^0_{nk} = {}_{m+1}X^1_{nk} = 0 \tag{2.3}$$

When writing (2.1) we took into account the fact that both forces may be applied to each spherical layer and that each force may give rise to its own stress and displacement field. Transformants (1.8), (1.10), (1.6) and (1.11), corresponding to this field, will be denoted by ${}_iu^0_{nk}$, ${}_iz^0_{nk}$, ${}_iz^{*0}_{nk}$, ${}_i\sigma^0_{rnk}$, ${}_i\tau^0_{nk}$, ${}_i\tau^{*0}_{nk}$. Since we can assume that this field arises in an unbounded medium with constants μ_i and G_i, the components of this field can always be determined using well-known formulae of the theory of elasticity, and we will therefore assume the transformants mentioned to be known. Hence, we need to determine the constants ${}_iX_{nk}$, ${}_iX^{0,1}_{nk}$, ${}_iY_{nk}$, ${}_iY^{0,1}_{nk}$ ($i = 0, 1, \ldots, m + 1$). Because of the introduction of the functions $z(r, \theta, \varphi)$, $z^*(r, \theta, \varphi)$, $\tau(r, \theta, \varphi)$, $\tau^*(r, \theta, \varphi)$ this problem splits into the problem of finding ${}_iX_{nk}$, ${}_iY_{nk}$ and ${}_iX^{0,1}_{nk}$, ${}_iY^{0,1}_{nk}$ separately.

We will write the method of solving this problem initially as it applies to ${}_iX_{nk}$, ${}_iY_{nk}$. We must primarily ensure that the displacements and stresses are continuous for $r = R_i$ ($i = 0, 1, \ldots, m$). In this case we are considering the stresses $\tau_{r\theta}$ and $\tau_{r\varphi}$, in terms of which τ^* is expressed, and the displacements u_θ and u_φ, which define z^*. This leads to the need to equate the function z_{nk} on the ith layer with $r = R_i$, divided by $2G_i$, to the analogous value of the same function on the $(i+1)$th layer, divided by $2G_{i+1}$. The continuity of the stresses $\tau_{r\theta}$ and $\tau_{r\varphi}$ with $r = R_i$ leads to an analogous operation with the function τ^*.

This holds provided that there is no defect (a crack or an inclusion) in the elastic medium when $r = R_i$ ($i = 0, 1, \ldots, m$). Since we are proposing to consider the case when there is a defect on the spherical surface $r = R_i$ in the section $l_0(\omega_1 \leqslant \theta \leqslant \omega_2)$, we need to introduce the jumps

$$ \begin{gathered} (2G_i)^{-1}z^*_n(R_i-0,\theta)-(2G_{i+1})^{-1}z^*_n(R_i+0,\theta)=\langle z^*_n(R_i,\theta)\rangle \\ \tau^*_n(R_i-0,\theta)-\tau^*_n(R_i+0,\theta)=\langle\tau^*_n(R_i,\theta)\rangle,\quad \theta\in l_0 \end{gathered} \tag{2.4}$$

and their Legendre transformants

$$ \int_{l_0} \sin t \begin{Vmatrix} \langle z^*_n(R_i t)\rangle \\ \langle \tau^*_n(R_i,t)\rangle \end{Vmatrix} P_k^{|n|}(\cos t)dt = \begin{Vmatrix} {}_iz^{*1}_{nk} \\ {}_i\tau^{*1}_{nk} \end{Vmatrix},\quad i=0,1,\ldots,m \tag{2.5}$$

We will write the condition for the displacements and the stresses to be continuous when $r = R_i$ ($i = 0, 1, \ldots, m$), taking into account the presence of the jumps (2.4) and (2.5), in the form

$$ \begin{gathered} \frac{{}_iX_{nk}R_i^k}{2G_i}+\frac{{}_iY_{nk}R^{-k-1}}{2G_i}-\frac{{}_{i+1}X_{nk}}{2G_{i+1}}-\frac{{}_{i+1}Y_{nk}R_i^{-k-1}}{2G_{i+1}}={}_iZ^*_{nk} \\ {}_iX_{nk}(k-1)R_i^{k-1}-{}_iY_{nk}(k+2)R_i^{-k-2}-{}_{i+1}X_{nk}(k-1)R_i^{k-1}+ \\ +{}_{i+1}Y_{nk}(k+2)R_i^{-k-2}=2{}_iT^*_{nk},\quad (i=0,1,\ldots,m) \end{gathered} \tag{2.6}$$

Here

$$ \begin{gathered} {}_iZ^*_{nk}={}_iz^{*1}_{nk}+{}_{i+1}z^{*0}_{nk}(R_i)(2G_{i+1})^{-1}-{}_iz^{*0}_{nk}(R_i)(2G_i)^{-1} \\ {}_iT^*_{nk}={}_i\tau^{*1}_{nk}+{}_{i+1}\tau^{*0}_{nk}(R_i)-{}_i\tau^{*0}_{nk}(R_i) \end{gathered}$$

In order to determine the coefficients ${}_iX_{nk}$ and ${}_iY_{nk}$ from (2.6), we will introduce the vectors

$$ \mathbf{x}_i=\begin{Vmatrix} {}_iX_{nk} \\ {}_iY_{nk} \end{Vmatrix},\quad \mathbf{f}_i=2\begin{Vmatrix} {}_iZ^*_{nk} \\ {}_iT^*_{nk} \end{Vmatrix},\quad i=0,1,\ldots,m+1 \tag{2.7}$$

which enables Eqs (2.6) to be written in the form

$$a_i\mathbf{x}_i - b_i\mathbf{x}_{i+1} = \mathbf{f}_i, \quad \mathbf{x}_{i+1} = c_i\mathbf{x}_i - b_i^{-1}\mathbf{f}_i; \quad c_i = b_i^{-1}a_i \tag{2.8}$$

$$a_i = \begin{Vmatrix} R_i^k G_i^{-1} & R_i^{-k-1}G_i^{-1} \\ (k-1)R_i^{k-1} & -(k+2)R_i^{-k-2} \end{Vmatrix}, \quad b_i = \begin{Vmatrix} R_i^k G_{i+1}^{-1} & R_i^{-k-1}G_{i+1}^{-1} \\ (k-1)R_i^{k+1} & -(k+2)R_i^{-k-2} \end{Vmatrix}$$

Using representation (2.8), the solution of Eqs (2.6) must be obtained in the form

$$\mathbf{x}_j = C_{j-1}^{(0)}\mathbf{x}_0 - \sum_{l=0}^{j-1} C_{j-1}^{(l+1)} b_l^{-1}\mathbf{f}_l, \quad i = \overline{0,m} \tag{2.9}$$

Here

$$C_j^{(l)} = c_j c_{j-1}\ldots c_l, \quad l < j; \quad C_j^{(l)} = c_j, \quad l = j; \quad C_j^{(l)} = I, \quad l < j \tag{2.10}$$

where I is the 2×2 identity matrix.

Here, by (2.3)

$$\mathbf{x}_0 = \begin{Vmatrix} {}_0X_{nk} \\ 0 \end{Vmatrix}, \quad \mathbf{x}_{m+1} = \begin{Vmatrix} 0 \\ {}_{m+1}Y_{nk} \end{Vmatrix} \tag{2.11}$$

In order to obtain the values of these vectors we put $i = m$ in (2.8) and substitute the expression for x_m, taken from (2.9) using (2.11). As a result we obtain

$$B_m \begin{Vmatrix} {}_0X_{nk} \\ 0 \end{Vmatrix} - b_m \begin{Vmatrix} 0 \\ {}_{m+1}Y_{nk} \end{Vmatrix} = \mathbf{f}_m + \sum_{l=0}^{m-1} D_l^{(m)}\mathbf{f}_l \tag{2.12}$$

where

$$B_m = a_m C_{m-1}^0 \begin{Vmatrix} B_{00}^{(m)} & B_{01}^{(m)} \\ B_{10}^{(m)} & B_{11}^{(m)} \end{Vmatrix}, \quad D_l^{(m)} = a_m C_{m-1}^{(l+1)} b_l^{-1} = \begin{Vmatrix} d_{00}^{l,m} & d_{01}^{l,m} \\ d_{10}^{l,m} & d_{11}^{l,m} \end{Vmatrix} \tag{2.13}$$

Solving system (2.12) we obtain

$$\begin{aligned} {}_0X_{nk} &= \Delta_m^{-1}[b_{11}^{(m)}F_{nk}^0 - b_{01}^{(m)}F_{nk}^1] \\ {}_{m+1}Y_{nk} &= \Delta_m^{-1}[B_{10}^{(m)}F_{nk}^0 - B_{00}^{(m)}F_{nk}^1] \\ \Delta m &= b_{11}^{(m)}B_{00}^{(m)} - b_{01}^{(m)}B_{10}^{(m)} \end{aligned} \tag{2.14}$$

Here, by (2.7), (2.8) and (2.12), (2.13) we have

$$\begin{aligned} b_{01}^{(m)} &= R_m^{-k-1}G_{m+1}^{-1}, \quad b_{11}^{(m)} = -(k+2)R_m^{-k-2} \\ F_{nk}^0 &= 2\left[mZ_{nk}^* + \sum_{l=0}^{m-1}({}_lZ_{nk}^* d_{00}^{l,m} + {}_lT_{nk}^* d_{01}^{l,m})\right] \\ F_{nk}^1 &= 2\left[mT_{nk}^* + \sum_{l=0}^{m-1}({}_lZ_{nk}^* d_{10}^{l,m} + {}_lT_{nk}^* d_{11}^{l,m})\right] \end{aligned} \tag{2.15}$$

Formulae (2.9), (2.11) and (2.14) completely define the unknown coefficients occurring in the expressions for the functions $z_{nk}^*(r)$ and $\tau_{nk}^*(r)$.

We will use this scheme to determine the remaining coefficients ${}_iX_{nk}^{0,1}$ and ${}_iY_{nk}^{0,1}$ $(i = 0, 1, \ldots, m+1)$, for which we will write the condition for the functions $u_{nk}(r)$, $z_{nk}(r)$ and $\sigma_{mk}(r)$, $\tau_{nk}(r)$ to be continuous on each spherical surface $r = R_i$ $(i = 0, 1, \ldots, m)$, taking into account the presence of the defect, i.e. involving the Legendre transformants ${}_iz_{nk}^1$, ${}_i\tau_{nk}^1$ of the jumps

$$\begin{aligned} z_n(R_i - 0,\theta)(2G_i)^{-1} - z_n(R_i + 0,\theta)(2G_{i+1})^{-1} &= \langle z_n(R_i,\theta)\rangle \\ \tau_n(R_i - 0,\theta) - \tau_n(R_i + 0,\theta) &= \langle \tau_n(R_i,\theta)\rangle, \quad \theta \in l_0 \end{aligned} \tag{2.16}$$

defined by formulae similar (2.5).

Introducing the vectors

$$
{}_i\mathbf{X}_{nk},\ {}_i\mathbf{Y}_{nk},\ {}_i\mathbf{V}_{nk},\ {}_i\mathbf{S}_{nk} = \left\|\begin{matrix} {}_iX^0_{nk} \\ {}_iX^1_{nk} \end{matrix}\right\|,\quad \left\|\begin{matrix} {}_iY^0_{nk} \\ {}_iY^1_{nk} \end{matrix}\right\|,\quad \left\|\begin{matrix} {}_iU_{nk} \\ {}_iZ_{nk} \end{matrix}\right\|,\quad \left\|\begin{matrix} {}_i\Sigma_{nk} \\ {}_iT_{nk} \end{matrix}\right\| \tag{2.17}
$$

$$
\begin{aligned}
{}_iU_{nk} &= {}_iu^1_{nk} + {}_{i+1}u^0_{nk}(R_i)(2G_{i+1})^{-1} - {}_iu^0_{nk}(R_i)(2G_i)^{-1} \\
{}_iZ_{nk} &= {}_iz^1_{nk} + {}_{i+1}z^0_{nk}(R_i)(2G_{i+1})^{-1} - {}_iZ^0_{nk}(R_i)(2G_i)^{-1} \\
{}_i\Sigma_{nk} &= {}_i\sigma^1_{rnk} + {}_{i+1}\sigma^0_{rnk}(R_i) - {}_i\sigma^0_{rnk}(R_i) \\
{}_iT_{nk} &= {}_i\tau^1_{nk} + {}_{i+1}\tau^0_{nk}(R_i) - {}_i\tau^0_{nk}(R_i)
\end{aligned}
$$

we can write the displacement continuity conditions

$$
\begin{gathered}
\alpha^{(i)}{}_i\mathbf{X}_{nk} - \alpha^{(i)}_*{}_{i+1}\mathbf{X}_{nk} + \beta^{(i)}{}_i\mathbf{Y}_{nk} - \beta^{(i)}_*{}_{i+1}\mathbf{Y}_{nk} = {}_i\mathbf{V}_{nk},\quad i = 0,1,\dots,m \\
\alpha^{(i)} = \frac{R_i^{k-1}}{2G_i}\left\|\begin{matrix} {}_i\mu^+_k R_i^2 & 1 \\ -{}_i\mu^-_{k+2}kR_i^2 & -k+1 \end{matrix}\right\|,\quad \alpha^{(i)}_* = \frac{R_i^{k-1}}{2G_{i+1}}\left\|\begin{matrix} {}_{i+1}\mu^+_k R_i^2 & 1 \\ -{}_{i+1}\mu^-_{k+2}R_i^2 k & -k-1 \end{matrix}\right\| \\
\beta^{(i)} = \frac{R_i^{-k-2}}{2G_i}\left\|\begin{matrix} {}_i\mu^-_k R_i^2 & -1 \\ {}_i\mu^+_{k-2}(k+1)R_i^2 & -k \end{matrix}\right\|,\quad \beta^{(i)}_* = \frac{R_i^{-k-2}}{2G_{i+1}}\left\|\begin{matrix} {}_{i+1}\mu^-_k R_i^2 & -1 \\ {}_{i+1}\mu^+_{k-2}(k+1)R_i^2 & -k \end{matrix}\right\|
\end{gathered} \tag{2.18}
$$

and the stress continuity conditions

$$
\begin{gathered}
\gamma^{(i)}{}_i\mathbf{X}_{nk} - \gamma^{(i)}_*{}_{i+1}\mathbf{X}_{nk} + \delta^{(i)}{}_i\mathbf{Y}_{nk} - \delta^{(i)}_*{}_{i+1}\mathbf{Y}_{nk} = {}_i\mathbf{S}_{nk},\quad i = 0,1,\dots,m \\
\frac{\gamma^{(i)}}{R_i^{k-2}} = \left\|\begin{matrix} {}_i\mu^\sigma_k R_i^2 & k-1 \\ -{}_i\mu_{k+1}kR_i^2 & 1-k^2 \end{matrix}\right\|,\quad \frac{\gamma^{(i)}}{R_i^{k-2}} = \left\|\begin{matrix} {}_{i+1}\mu^\sigma_k R_i^2 & k-1 \\ -{}_{i+1}\mu^\tau_{k+1}R_i^2 & 1-k^2 \end{matrix}\right\| \\
-\frac{\delta^{(i)}}{R_i^{k-3}} = \left\|\begin{matrix} {}_i\mu^\sigma_{k+2}R_i^2 & -k-2 \\ {}_i\mu^\tau_k(k+1)R_i^2 & -k(k+2) \end{matrix}\right\|,\quad \frac{\delta^{(i)}}{R_i^{k-3}} = \left\|\begin{matrix} {}_{i+1}\mu^\sigma_{k+1}R_i^2 & -k-2 \\ {}_{i+1}\mu^\tau_k(k+1)R_i^2 & -k(k+2) \end{matrix}\right\|
\end{gathered} \tag{2.19}
$$

We can reduce the system of equations (2.18), (2.19) to Eq. (2.8), already investigated, if we introduce four-dimensional vectors and the corresponding matrices

$$
\mathbf{x}_i = \left\|\begin{matrix} {}_i\mathbf{X}_{nk} \\ {}_i\mathbf{Y}_{nk} \end{matrix}\right\|,\quad \mathbf{f}_i = \left\|\begin{matrix} {}_i\mathbf{V}_{nk} \\ {}_i\mathbf{S}_{nk} \end{matrix}\right\|;\quad a_i = \left\|\begin{matrix} \alpha^{(i)} & \beta^{(i)} \\ \gamma^{(i)} & \delta^{(i)} \end{matrix}\right\|,\quad b_i = \left\|\begin{matrix} \alpha^{(i)}_* & \beta^{(i)}_* \\ \gamma^{(i)}_* & \delta^{(i)}_* \end{matrix}\right\| \tag{2.20}
$$

Consequently, the solution of system (2.18), (2.19) can be written in the form (2.9), but the vectors and matrices must be taken as in (2.20). Here formulae (2.11) still hold, only instead of ${}_0X_{nk}$ and ${}_{m+1}Y_{nk}$ we must take the vectors ${}_0\mathbf{X}_{nk}$, ${}_{m+1}\mathbf{Y}_{nk}$. It can be shown that formulae (2.14) hold for determining them with the following correction: ${}_0\mathbf{X}_{nk}$ and ${}_{m+1}\mathbf{Y}_{nk}$ must be replaced by $\mathbf{X}_{nk}$ and ${}_{m+1}\mathbf{Y}_{nk}$, while F^0_{nk} and F^1_{nk} must be replaced by

$$
\begin{aligned}
\mathbf{F}^0_{nk} &= {}_m\mathbf{V}_{nk} + \sum_{l=0}^{m-1}(d^{l,m}_{00}{}_l\mathbf{V}_{nk} + d^{l,m}_{01}{}_l\mathbf{S}_{nk}) \\
\mathbf{F}^1_{nk} &= {}_m\mathbf{S}_{nk} + \sum_{l=0}^{m-1}(d'^{l,m}_{10}{}_l\mathbf{V}_{nk} + d'^{l,m}_{11}{}_l\mathbf{S}_{nk})
\end{aligned} \tag{2.21}
$$

respectively, where in the representations of the matrices (2.13) their components $B^{(m)}_{jk}$ and $d^{l,m}_{jk}$ are 2×2 partitioned matrices and, in particular, the numbers b^m_{11} and b^m_{01} must be replaced by the matrices $\delta^{(m)}_*$ and $\beta^{(m)}_*$, respectively.

Moreover, in the formula for ${}_{m+1}\mathbf{Y}_{nk}$ the matrix Δ_m must be replaced by the matrix $\Delta m = B^{(m)}_{00}\delta^{(m)}_* - B^{(m)}_{10}\beta^{(m)}_*$. Hence, all the coefficients ${}_iX_{nk}$, ${}_iY_{nk}$, ${}_iX^{0,1}_{nk}$, ${}_iY^{0,1}_{nk}$ are obtained, and knowing them we can determine the transformants (2.1) and (2.2).

The solution of this problem will be completed if we obtain the jumps (2.4) and (2.16). To obtain the corresponding equations we need to substitute the coefficients obtained into (2.1) and into the similar formulae for (2.2) and invert the Legendre transformants obtained. By subsequently satisfying the conditions at the defect in the Fourier transformants we can obtain integral and integro-differential equations for determining these jumps.

In order not to obscure the matter with lengthy calculations, we will carry out these operations in a special case of the problem under discussion.

3. REDUCTION OF THE PROBLEM OF THE STRESS CONCENTRATION NEAR A CRACK WHEN ACTED UPON BY A TORSION CENTRE TO AN INTEGRAL EQUATION

We will assume that the spherically multilayered medium described above is subjected to a torsion centre with torque M at the origin of coordinates. In view of the axial symmetry (the required and specified functions are independent of φ) we must put $n = 0$ in all the formulae presented above. The stress and displacement fields will be determined solely by the functions $\tau_{r\varphi}$, u_φ and, by (1.1) and (1.9), by the functions

$$\left\|\begin{matrix}\tau_0^*(r,\ \theta)\\ z_0^*(r,\ \theta)\end{matrix}\right\| = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left\|\begin{matrix}\tau_{r\theta}(r,\ \theta)\\ 2Gu_\varphi(r,\ \theta)\end{matrix}\right\| \tag{3.1}$$

respectively.

In the case of the loading considered, in relations (2.6), by (1.14), we must take

$$_0\tau_0^{*0}(r,\ \theta) = \frac{3M\cos\theta}{4\pi r^3},\quad {}_0z_0^{*0}(r,\ \theta) = -\frac{M\cos\theta}{2\pi r^2},\quad {}_i\tau_0^{*0}(r,\ \theta) = {}_iz_0^{*0}(r,\ \theta) = 0 \tag{3.2}$$

$$i = 1,\ 2, \ldots m$$

and therefore

$$_0\tau_{0k}^{*0}(R_0) = \frac{M\delta_{k,1}}{2\pi R_0^3},\quad {}_0z_{0k}^{*0}(R_0) = -\frac{M\delta_{k,1}}{3\pi R_0^2},\quad {}_i\tau_{0k}^{*0} = {}_iz_{0k}^{*0} = 0\quad i = 1, 2, \ldots,\ m \tag{3.3}$$

To solve this problem we must first solve the system of equations (2.6) using (2.9) and (2.14). To fix our ideas we will confine ourselves to the case of a two-layer medium ($m = 0$) and we will assume that there is an interphase crack at $r = R_0$, i.e. $l_0 = [0, \omega]$ in (2.4) and (2.5).

If we assume that the sides of the crack are not loaded, we have $\langle\tau_0^*(R_0, \theta)\rangle = 0$, $_0\tau_{0k}^{*1} = 0$ and therefore, by (2.6) and (2.15)

$$F_{0k}^0 = 2[{}_0z_{0k}^{*1} - (2G_0)^{-1}{}_0z_{0k}^{*0}(R_0)],\quad F_{0k}^1 = -{}_0\tau_{0k}^{*0}(R_0)2 \tag{3.4}$$

Moreover, by (2.10) $C_{-1}^0 = I$ and hence $B_0 = a_0$.

In this case, of all the coefficients in Eqs (2.6) we must only obtain $_0X_{0k}$ and $_1Y_{0k}$. We find them using Eqs (2.14), taking (3.4) and (3.3) into account. We obtain

$$_0X_{0k} = \frac{2(k+2)G_0\ {}_0z_{0k}^{*1} + g_k^0}{R_0^k\gamma_k},\quad {}_1Y_{0k} = -\frac{2(k-1)G_0\ {}_0z_{0k}^{*1} + g_k^1}{R_0^{-k-1}\gamma_k} \tag{3.5}$$

$$[q_k^0,\ q_k^1] = \frac{M\delta_{k1}}{3\pi R_0^2}[k+2-3\gamma,\ k+2],\quad \gamma_k = 2-\gamma+(1+\gamma)k,\quad \gamma = \frac{G_0}{G_1}$$

Substituting these expressions into the corresponding formulae for $_iz_{0k}^*(r)$ and $_i\tau_{0k}^*(r)$ we obtain from (2.2), with $i = 0$ and $i = 1$ ($n = 0$) and taking (2.3) into account, the Legendre transformants of the functions $z_0^*(r, \theta)$ and $\tau_0^*(r, \theta)$ when $0 < r < R_0$ ($i = 0$) and $R_0 < r < \infty$ ($i = 1$). Subsequent inversion using the appropriate formula from (1.4) with $n = 0$, enables us to obtain the values of these functions.

For example, for $\tau_0^*(r, \theta)$ when $0 < r < R_0$ we will have (using (3.2))

$$r\tau_0^*(r,\ \theta)=G_0\int_0^\omega\left\langle z_0^*(R_0,\ t)\right\rangle\sin t\sum_{k=0}^\infty\left(\frac{r}{R_0}\right)^k\frac{(k-1)(k+2)P_k(\cos\theta)P_k(\cos t)}{2\gamma_k(2k+1)^{-1}}dt+$$
$$+2M(4\pi r^2)^{-1}\cos\theta^0 \tag{3.6}$$

In order to obtain the equation for determining the jump

$$\left\langle z_0^*(R_0t)\right\rangle=\frac{2G_0}{\sin t}\frac{\partial}{\partial t}\sin t\left\langle u_\varphi(R_0t)\right\rangle=\chi(t) \tag{3.7}$$

(the first equations follows from (3.1)) we must implement the condition at the defect (in this case, a crack): the stress $\tau_{r\varphi}(r,\theta)$ is equal to zero on the sides of the crack $r = R_0 - 0$ and $r = R_0 + 0$, which means that we must do the same for $\tau_0^*(r,\theta)$. Since the condition $\langle\tau_0^*(R_0,\theta)\rangle \equiv 0$ has already been used to obtain (3.5), it is sufficient to implement the condition

$$\overset{*}{\tau}_0(R_0^{-1},\theta)=0,\qquad 0\leqslant\theta\leqslant\omega \tag{3.8}$$

Substituting (3.6) into (3.8), we obtain the necessary equation for determining the required jump (3.7). In order to reduce it to a simpler form, we must take into account the fact that

$$(k+2)(k-1)=k(k+1)-2,\quad \nabla_0P_k(\cos\theta)=k(k+1)P_k(\cos\theta)$$

This enables the equation obtained to be written in the form

$$(\nabla_0-2)Y_0^*(\theta)=-A\cos\theta,\quad 0\leqslant\theta\leqslant\omega;\quad A=3M(4\pi R_0^2G_0)^{-1} \tag{3.9}$$
$$Y_0^*(\theta)=\int_0^\omega\chi(t)\sin tS_0^*(\theta,\ t)dt,\quad S_0^*(\theta,\ t)=\sum_{k=0}^\infty\frac{2k+1}{\gamma_k}P_k(\cos\theta)P_k(\cos t)$$

In order to convert the integro-differential equation obtained into an integral equation, we carry out the following operations. Bearing in mind that

$$(\nabla_0-2)y(\theta)\equiv-ly(\theta),\quad ly(\theta)=y''(\theta)+\operatorname{ctg}\theta y'(\theta)-2y(\theta) \tag{3.10}$$

and $lP_1(\cos\theta) = lQ_1(\cos\theta) = 0$ ($Q(z)$ is a Legendre function of the second kind), we write the general solution of the differential equation corresponding to (3.9) and (3.10), regular at zero, in the form

$$Y_0^*(\theta)=A\int_0^\theta\cos tK(\theta,\ t)dt+\frac{C_1}{3\sqrt{\pi}}P_1(\cos\theta)\equiv f^*(\theta) \tag{3.11}$$

where C_1 is an arbitrary constant, and its fundamental function (solution) has the form

$$\sqrt{\pi}K(\theta,\ t)=2\sin t[P_1(\cos\theta)Q_1(\cos t)-P_1(\cos t)Q_1(\cos\theta)]$$
$$P_1(\cos\theta)=\cos\theta,\quad Q_1(\cos\theta)=\cos\theta\ln\operatorname{ctg}{\textstyle\frac12}\theta-1$$

Hence, the integral equation of the problem considered can be finally written in the form

$$\int_0^\omega\chi(t)\sin t\,S_0^*(\theta,\ t)dt=f^*(\theta),\quad 0\leqslant\theta\leqslant\omega \tag{3.12}$$
$$3\sqrt{\pi}f^*(t)=(C_1+2A)\cos\theta-2A(1-2\cos\theta\ln\cos{\textstyle\frac12}\theta)$$

(the expression for the right-hand side is obtained after evaluating the integrals in (3.11)). The arbitrary constant C_1 which occurs here is obtained from the condition for the crack to be closed, which, by virtue of (3.7), can be written as

$$\int_0^\omega\sin\theta\chi(\theta)d\theta=2G[\sin\theta\left\langle u_\varphi(R_0\theta)\right\rangle]_0^\omega=0 \tag{3.13}$$

4. SOLUTION OF THE INTEGRAL EQUATION

We first separate the irregular (weakly polar) part from the kernel (3.9) of integral equation (3.12). To do this we use the easily verified relation

$$S_0^*(\theta,\ t)=2(\gamma+1)^{-1}[S_0(\theta,\ t)+\tfrac{3}{2}(\gamma-1)S_0^1(\theta,\ t)]$$

where

$$S_0(\theta,\ t)=\frac{1}{2}\sum_{k=0}^{\infty}P_k(\cos\theta)P_k(\cos t),\quad S_0^1(\theta,\ t)=\sum_{k=0}^{\infty}\frac{P_k(\cos\theta)P_k(\cos t)}{2\gamma_k} \tag{4.1}$$

The first of these series (the weakly polar part of the kernel) is added [6] to the discontinuous Weber–Sonin integral, i.e.

$$S_0(\theta,\ t)=\frac{W_0(\operatorname{tg}\tfrac{1}{2}\theta,\ \operatorname{tg}\tfrac{1}{2}t)}{2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}t},\quad W_0(x,\ y)=\int_0^{\infty}J_0(tx)f_0(ty)dt \tag{4.2}$$

However, the second series from (4.1) will also not be a continuous function. In order to convince ourselves of this and to separate the discontinuous part, we note the well-known relation

$$\sum_{k=1}^{\infty}\frac{2k+1}{2k(k+1)}P_k(\cos\theta)P_k(\cos t)=-\frac{1}{2}-\ln\sin\frac{\theta}{2}-\ln\cos\frac{t}{2}$$

Then, if we take into account the directly verifiable equations

$$\frac{1}{\gamma_k}=\frac{1}{(\gamma+1)(k+\beta)},\quad \beta=\frac{2-\gamma}{1+\gamma},\quad \frac{1}{k+\beta}-\frac{2k+1}{2k(k+1)}=\frac{1}{k}\left(\frac{1}{2(k+1)}-\frac{\beta}{k+\beta}\right)$$

it can be shown that

$$S_0^1(\theta,\ t)=\frac{1}{2(\gamma+1)}\left[\frac{1+2\gamma}{2(2-\gamma)}-\ln\sin\frac{\theta}{2}+R_0(\theta,\ t)\right] \tag{4.3}$$

$$R_0(\theta,\ t)=\sum_{k=0}^{\infty}\frac{\beta_k}{k}P_k(\cos\theta)P_k(\cos t),\quad \beta_k=\frac{1}{2k+1}-\frac{\beta}{k+\beta}$$

The last function will be continuous.

Taking (4.1) and (4.3) into account and also condition (3.13) for the crack to be closed, integral equation (3.12) can be written in the form

$$L\chi\equiv\int_0^{\omega}[S_0(\theta,\ t)+\lambda R_0(\theta,\ t)]\sin t\,\chi(t)dt=\sum_{i=0}^{1}C_1^*P_i(\cos\theta)-$$

$$-g_2(\theta),\ 0\leqslant\theta\leqslant\omega;\quad 4\lambda=3(\gamma-1)(\gamma+1)^{-1} \tag{4.4}$$

$$g_2(\theta)=\frac{A(\gamma+1)}{3\sqrt{\pi}}\left(1+2\cos\theta\ln\cos\frac{\theta}{2}\right),\quad C_0^*=\lambda\int_0^{\omega}\sin t\,\chi(t)\ln\cos\frac{t}{2}$$

where C_1^* is a new arbitrary constant, related to C_1.

According to the structure of the right-handside, the solution of Eq. (4.4) must be constructed in the form of a series

$$\chi(\theta)=\sum_{i=0}^{1}C_i^*\chi_i(\theta)-\chi_2(\theta) \tag{4.5}$$

each term of which satisfies one of the equations

$$L\chi_i(\theta)=g_i(\theta),\quad 0\leqslant\theta\leqslant\omega,\quad i=0,\ 1,\ 2;\quad g_i(\theta)=P_i(\theta),\quad i=0,\ 1 \tag{4.6}$$

If these equations are solved, the constants C_0^* and C_1^* are found by implementing condition (3.13) and the last equation of (4.4).

In order to reduce Eq. (4.6) to a known form, we make the replacements

$$\operatorname{tg}\frac{\theta}{2}=ax,\quad \operatorname{tg}\frac{t}{2}=ay,\quad a=\operatorname{tg}\frac{\omega}{2} \tag{4.7}$$

$$X_i(y)=\frac{2a\chi(2\operatorname{arctg}ay)}{[1+(ay)^2]^{3/2}},\quad F_i(x)=\frac{q_i(2\operatorname{arctg}ax)}{[1+(ax)^2]^{1/2}}$$

Then, instead of (4.6), we will have

$$\int_0^1[W_0(x,\ y)+\lambda D_0(x,\ y)]yX_i(y)dy=F_i(x),\quad 0\leqslant x\leqslant 1 \tag{4.8}$$

$$D_0(x,\ y)=[2a\cos\frac{\theta}{2}\cos\frac{t}{2}R_0(\theta,\ t)]_{\theta=2\operatorname{arctg}ax,t=2\operatorname{arctg}ay}$$

Integral equation (4.8) has already been encountered in contact problems [8]. To solve it approximately it is convenient to use the method of orthogonal polynomials [8], in view of the presence of the spectral relation A5.2 from [8], according to which, the solution of Eq. (4.8) is constructed in the form ($P_k^{\alpha,\beta}(z)$ is a Jacobi polynomial)

$$X_i(y)=\sum_{k=0}^{\infty}\frac{X_k^iP_k^{0-1/2}(1-2y^2)}{\sqrt{1-y^2}},\quad P_k^{0,-1/2}(1-2y^2)=P_{2k}(\sqrt{1-y^2})$$

The next step in the method of orthogonal polynomials [7], as it applies to this equation, reduces it to an infinite system

$$Y_j^{(i)}+\lambda\sum_{k=0}^{\infty}d_{jk}Y_k^{(i)}=F_j^{(i)},\quad i=0,\ 1,\ 2;\ Y_j^{(i)}=\nu_jX_j^{(i)},\quad \nu_j=\frac{\Gamma(j+1/2)}{j\sqrt{2(2j+1)}} \tag{4.9}$$

$$F_j^{(i)}=\int_0^{\omega}\frac{\operatorname{tg}{1/2}\theta g_i(\theta)Q_j^*(\theta)}{2\nu_ja^2\cos{1/2}\theta}d\theta,\quad Q_j^*(\theta)=\frac{P_{2j}(\sqrt{1-a^{-2}(\operatorname{tg}{1/2}\theta)^2})}{\sqrt{1-a^{-2}(\operatorname{tg}{1/2}\theta)^2}}$$

$$d_{jk}=\iint_0^{\omega}\frac{R_0(\theta,\ t)\operatorname{tg}{1/2}\theta\operatorname{tg}{1/2}tQ_j^*(\theta)Q_k^*(t)}{2\nu_j\nu_ka^3\cos{1/2}\theta\cos{1/2}t}d\theta dt$$

The infinite system obtained must be solved approximately by the reduction method; the convergence of this method can be proved using the scheme described earlier in [8].

If the infinite systems (4.9) are solved for $i = 0, 1, 2$ and the constants C_0 and C_1 are obtained from the conditions indicated above, the solution of the equation under discussion is obtained from (4.5) or, by (4.7) and (4.9), from the formula

$$X(x)=\frac{2a\chi(2\operatorname{arctg}ax)}{[1+(ax)^2]^{3/2}}=\sum_{k=0}^{\infty}\frac{X_kP_k^{0-1/2}(1-2x^2)}{\sqrt{1-x^2}} \tag{4.10}$$

$$X_k=C_0^*X_k^0+C_1^*X_k^{(1)}-X_k^{(2)}$$

5. CALCULATION OF THE STRESS INTENSITY FACTOR

For the problem in question the stress intensity factor is found from the formula

$$K_{\mathrm{III}}=\lim_{\theta\to\omega+0}\tau_{r\varphi}(R_0\theta)\sqrt{2\pi R_0(\theta-\omega)} \tag{5.1}$$

Taking relation (3.1) into account we obtain

$$\tau_{r\varphi}(R_0,\ \theta)=\frac{1}{\sin\theta}\int_0^{\theta}\sin t\tau_0^*(R_0,\ t)dt \tag{5.2}$$

Using (3.7) and (3.9) we obtain from (3.6)

$$R_0\tau_0^*(R_0-0,\theta)=G_0[(\nabla_0-2)Y_0^*(\theta)+A\cos\theta],\quad 0\leqslant\theta\leqslant\pi \tag{5.3}$$

In view of the fact that

$$\frac{1}{\sin\theta}\int_0^\theta \sin t\nabla_0 Y_0^*(t)dt=\frac{dY_0^*(\theta)}{d\theta}$$

formula (5.1) can be written, using (5.2), in the form

$$K_{\rm III}=-\sqrt{\frac{2\pi}{R_0}}G_0\lim_{\theta\to\omega+0}\sqrt{\theta-\omega}\frac{d}{d\theta}Y_0^*(\theta) \tag{5.4}$$

When obtaining this formula we took into account the fact that all the functions which occur as terms in the formula for $\tau_{r\varphi}(R_0,\theta)$, which have a finite limit as $\theta\to\omega+0$, drop out. On the same basis, by virtue of (3.9), (4.1) and (4.3), instead of (5.4) we can write

$$K_{\rm III}=-\sqrt{\frac{2\pi}{R_0}}\lim_{\theta\to\omega+0}\frac{2\sqrt{\theta-\omega}}{\gamma+1}Y_0'(\theta),\quad Y_0(\theta)=\int_0^\infty\chi(t)\sin tS_0(\theta,\ t)dt \tag{5.5}$$

If we take (4.2) into account in the last integral and make replacement (4.7), we obtain the relation

$$Y_0(\theta)=Y_0(2\,\mathrm{arctg}\,ax)=\frac{\tilde{Y}_0(x)}{[1+(ax)^2]^{-1/2}},\quad \tilde{Y}_0(x)=\int_0^1W_0(x,y)X(y)ydy \tag{5.6}$$

Hence it follows that

$$2aY_0'(\theta)=[1+(ax)^2]^{3/2}\tilde{Y}_0'(x)+ax\sqrt{1+(ax)^2}\tilde{Y}_0(x)$$

and hence

$$K_{\rm III}=-\sqrt{\frac{8\pi}{R_0}}\frac{\sec\frac{1}{2}\omega}{\sin\omega}\frac{\tilde{A}N}{1+\gamma},\quad N=\lim_{x\to1+0}\sqrt{x^2-1}\tilde{Y}_0'(x) \tag{5.7}$$

$$\tilde{A}=\lim_{x\to1+0}\sqrt{\frac{2\,\mathrm{arctg}\,ax-2\,\mathrm{arctg}\,a}{x^2-1}}=\sqrt{\frac{\sin\omega}{2}}$$

In order to take the limit for N we must bear (5.6) and (4.10) in mind, which enables us to write

$$N=\sum_{k=0}^\infty X_k\lim_{x\to1+0}\frac{\Lambda_k'(x)}{(x^2-1)^{-1/2}},\quad \Lambda_k(x)=\int_0^1\frac{W_0(x,y)P_{2k}(\sqrt{1-y^2})dy}{y^{-1}\sqrt{1-y^2}}=$$

$$=\Gamma(\tfrac{1}{2}+k)[2\Gamma(\tfrac{3}{2}+2k)k!]^{-1}x^{-2k-1}F(\tfrac{1}{2}+k,\tfrac{1}{2}+k;\tfrac{3}{2}+2k;x^{-2}),\quad x>1 \tag{5.8}$$

The expression for the last integral in terms of the hypergeometric function is taken from [8]. Carrying out the differentiation using the well-known rules for the differentiation of hypergeometric functions and extending the result obtained analytically in the neighbourhood of unity, using formula (9.131.2) from [9] we obtain

$$\lim_{x\to1+0}\sqrt{x^2-1}\Lambda_k'(x)=-\frac{\sqrt{\pi}}{k!\Gamma(\frac{1}{2}+k)} \tag{5.9}$$

Using (5.7), (5.8) and (5.9) we finally obtain

$$K_{\rm III}=\frac{2\pi}{1+\gamma}\frac{\sec\frac{1}{2}\omega}{\sqrt{R_0\sin\omega}}\sum_{k=0}^\infty\frac{X_k}{\Gamma(\frac{1}{2}+k)k!}$$

If there are no layers, i.e. $G_0 = G_1, \gamma = 1, \lambda = 0$, integral equation (4.8) has an accurate solution, as was shown previously in [5]. In this case the infinite systems degenerate into explicit formulae

$$Y_k^{(i)} = F_j^{(i)} \ (i = 0, 1, 2; k = 0, 1, 2, \ldots)$$

6. CONCLUSION

The proposed method has been described as it applies to the case when the defects are situated on spherical surfaces where the elasticity constants change (interphase defects). To cover the case when there is no interphase defect, one can proceed in two ways: (1) introduce, in addition, two spherical layers with the same elasticity constants, on the adjacent boundary of which the defect is situated, and (2) together with the term which takes into account the action of the body forces, applied to the spherical layer, where the defect in question is situated, one can introduce a discontinuous solution of Lamé's equations for this defect. The second way has the advantage that it enables one to cover the case of a non-interphase defect of arbitrary shape.

The method can be extended fairly simply to the case of a bounded spherically multilayered medium. We will describe the additional operations which are necessary to do this.

Suppose the spherically multilayered medium considered fills the region $R_0 \leqslant r \leqslant R_m$ Suppose that the displacements are specified on the boundary $r = R_0$ and, consequently, their Fourier–Legendre transformants are specified also. We will denote this transformant of the function $z^*(r, \theta, \varphi)$ by A_{nk}. Then, instead of the displacement and stress continuity conditions in (2.6) at $r = R_0$ we must write the boundary condition

$$z_{nk}^*(R_0) = {}_1X_{nk}R_0^k + {}_1Y_{nk}R_0^{-k-1} + {}_1z_{nk}^{*0}(R_0) = A_{nk} \tag{6.1}$$

From this equation we obtain ${}_1Y_{nk}$ and hence, from (2.7) we can write

$$\mathbf{x}_1 = \begin{Vmatrix} 1 \\ -R_0^{2k+1} \end{Vmatrix} {}_1X_{nk} + R_0^{k+1} \begin{Vmatrix} 0 \\ 1 \end{Vmatrix} [A_{nk} - {}_1z_{nk}^{*0}(R_0)] \tag{6.2}$$

The following formula holds for the remaining required vectors

$$\mathbf{x}_j = C_{j-1}^{(1)}\mathbf{x}_1 + \sum_{l=1}^{j-1} C_{j-1}^{(l+1)} b_l^{-1}\mathbf{f}_l, \quad j = 1, 2, \ldots, m-1 \tag{6.3}$$

obtained in exactly the same way as (2.9).

If we assume that the stresses and, of course, their Fourier–Legendre transformants, including the transformants $\tau_{nk}(r)$ and $\tau_{nk}^*(r)$, are specified on the other boundary $r = R_m$ of the elastic medium, similar operations are carried out. Suppose, for example, we are given the latter, i.e. $\tau_{nk}^*(R_m) = B_{nk}$. Then, instead of the displacement and stress continuity conditions at $r = R_m$ in (2.6) we must satisfy the boundary condition

$$2\tau_{nk}^*(R_m) = {}_mX_{nk}(k-1)R_m^{k-1} - {}_mY_{nk}(k+2)R_m^{-k-2} + {}_m\tau_{nk}^{*0}(R_m)2 = 2B_{nk}$$

Using this, in the same way as for (6.2), we will have

$$\mathbf{x}_m = \begin{Vmatrix} R_m^{2k+1}(k+2)(k-1)^{-1} \\ 1 \end{Vmatrix} {}_mY_{nk} - \frac{2}{(k-1)R_m^{k-1}} \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} [B_{nk} - {}_m\tau_{nk}^{*0}(R_m)] \tag{6.4}$$

It only remains to obtain ${}_1X_{nk}$ and ${}_mY_{nk}$. This is done using the same operations as when obtaining ${}_0X_{nk}$ and ${}_0Y_{nk}$ from the system of equations (2.6), i.e. we assume $i = m - 1$ in (2.8) and substitute (6.4) there, while the expression for $\mathbf{x}_{m-1}$ is taken from (6.3). We thereby obtain two algebraic equations for finding ${}_1X_{nk}$ and ${}_mY_{nk}$.

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